Sydney Technical High School

Mathematics Extension 1

H.S.C ASSESSMENT TASK 2

MARCH 2012

General Instructions

- Working Time – 70 minutes.
- Approved calculators may be used.
- A table of Standard Integrals is provided at the back of this paper.
- All necessary working should be shown for every question.
- Begin each question on a new side of the answer booklet.
- Marks shown are a guide and may need to be adjusted.
- Full marks may not be awarded for careless work or illegible writing.

NAME ______________________

TEACHER ______________________

Question 1

a) Find : i) $\int (7x-2)^4\,dx$ 1

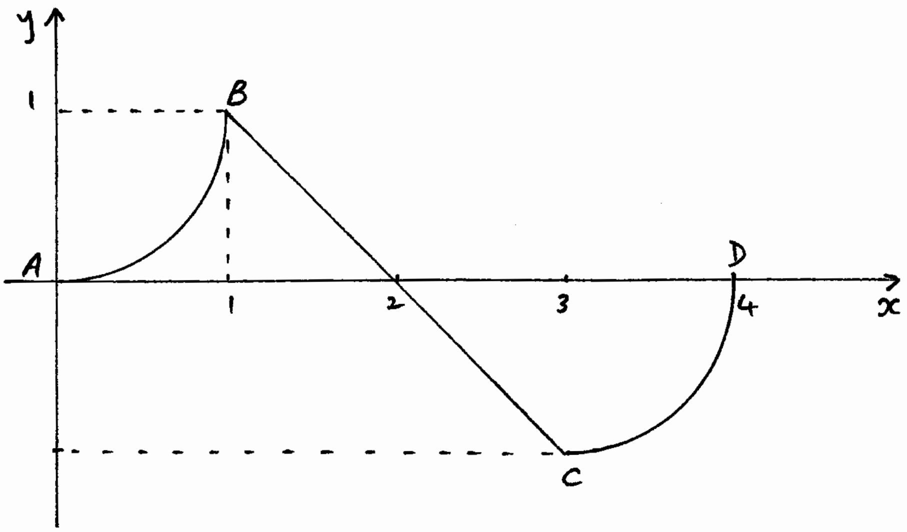
ii) $\int \frac{x+1}{\sqrt{x}}\,dx$ 2

b) Use the substitution $u = 2 + x^2$, or otherwise, to evaluate $\int_0^1 \frac{x}{(2+x^2)^2}\,dx$ 3

c) Solve $4\,cos^3x - 3\cos x = 0$ for $0 \le x \le 2\pi$ 3

Question 2 (Start a new page

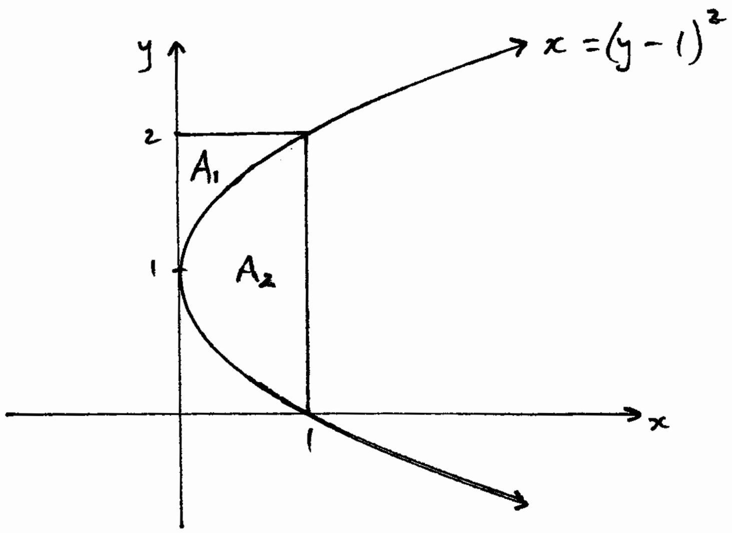
a) Below is shown the curve $y = f(x)$ for $0 \le x \le 4$. AB and CD are arcs of circles with centres (0,1) and (3,0) respectively.



i) Evaluate $\int_0^1 f(x)\,dx$ 1

ii) Evaluate $\int_0^4 f(x)\,dx$ 1

b) Shown is the parabola $x = (y-1)^2$



A_1 is the area bounded by the parabola, the y axis and $y = 2$.

A_2 is the area bounded by the parabola and $x = 1$.

i) Find A_1 1

ii) Find A_2 1

iii) A_2 is rotated about the y-axis. Find the volume thus generated, in exact form. 3

c) Evaluate $\int_{-2}^{2} \frac{x}{1+x^4}\, dx$ 1

Question 3 (Start a new page)

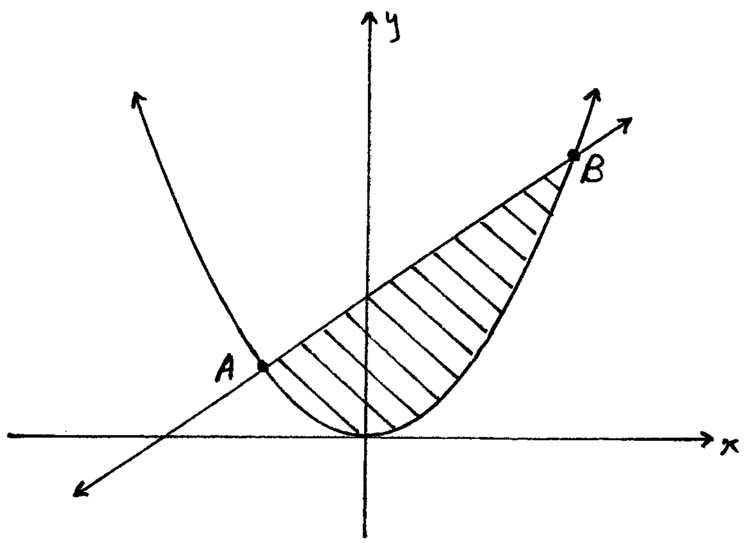
a) Find $\frac{d}{dx}[sin(\tan x)]$ 1

b) The data below gives values of $y = f(x)$ for $1 \le x \le 5$. 1

x	1	2	3	4	5
$f(x)$	0	1.4	3.3	2.8	1.5

Use the Trapezoidal Rule and 5 function values to approximate $\int_1^5 f(x)\, dx$.

c)



The area between the graphs of $y = x^2$ and $y = x + 2$ is shown. A and B are points of intersection of the two graphs.

i) Find x values for A and B. 1

ii) Find the value of the shaded area, correct to 1 dec. place 2

iii) The shaded area is rotated about the x-axis.

α) Write an integral expression to calculate the exact volume generated (do <u>not</u> evaluate). 1

β) Use Simpson's Rule and 3 function values to approximate the above volume. Leave your answer in simplest exact form. 3

Question 4 (Start a new page)

a) Simplify $\sin(\pi + \theta)\,\text{cosec}(\pi - \theta)$ 1

b) Given the curve represented by $y = 1 - x - \frac{1}{x-1}$.

i) Find y' and show that $y'' = \frac{-2}{(x-1)^3}$ 1

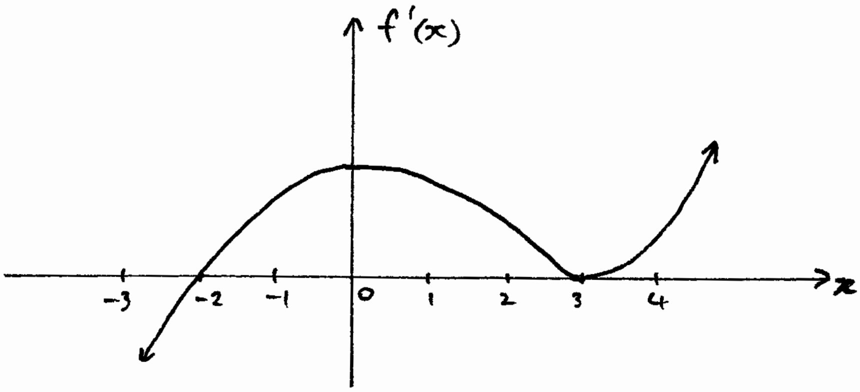
ii) Locate and determine the nature of any stationary points. 3

iii) Locate any points of inflexion. Give reasons. 1

iv) Draw a neat sketch of the curve. Use a ruler for straight lines and label key features. 2

Question 5 (Start a new page)

a)



The diagram above shows the graph of $y = f'(x)$, i.e. the derivative curve of $y = f(x)$.

i) Give the locations <u>and</u> types of the stationary points on the curve $y = f(x)$. 2

ii) Which feature will appear on the curve $y = f(x)$ that corresponds to $x = 0$ above? Justify. 1

iii) For what values of x is the curve $y = f(x)$: $\alpha)$ increasing ? 1

$\beta)$ such that $f''(x) < 0$ 1

iv) Neatly sketch a possible graph of $y = f(x)$, showing important x values. 1

v) Neatly sketch a possible graph of $y = f''(x)$, showing important x values. 1

b) i) Show that $\frac{d}{dx}(sec^2 2x) = 4 \tan 2x \, sec^2 2x$ 2

ii) Hence, find $\int_0^{\frac{\pi}{3}} \tan 2x \, sec^2 2x \; dx$ 1

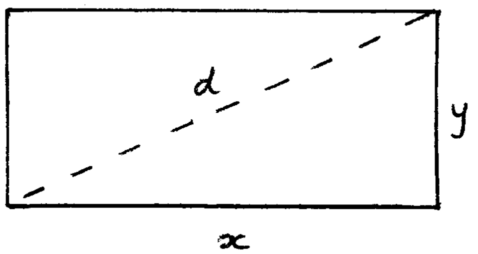
Question 6 (Start a new page)

a) i) On the same axes, neatly sketch and label the curves $y = \cos 3x$ 2

and $y = \cos 3\left(x - \frac{\pi}{6}\right)$ for $0 \leq x \leq \pi$. Clearly show intercepts on the axes.

ii) For what values of k will $\cos 3\left(x - \frac{\pi}{6}\right) = k$ have exactly 2 solutions, for $0 \leq x \leq \pi$? 1

b) A rectangle has fixed perimeter P cm. Its length, width and diagonal are variable and shown below. 4



Use calculus to prove that the shortest diagonal occurs when the rectangle is a square.

END OF TEST

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Solutions.

1 a) i) $\frac{(7x-2)^5}{35}+c$

ii) $\int\left(\frac{x}{\sqrt{x}}+\frac{1}{\sqrt{x}}\right)dx = \int\left(x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)dx$ ①

$= \frac{2}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}+c$ ①

b) $\int_0^1 \frac{x}{(2+x^2)^2}\,dx = \int_2^3 \frac{x}{u^2}\frac{du}{2x}$

$u = 2+x^2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$x=0,\ u=2$

$x=1,\ u=3$ ①

$= \frac{1}{2}\int_2^3 u^{-2}\,du$ ①

$= -\frac{1}{2}\left[\frac{1}{u}\right]_2^3$

$= -\frac{1}{2}\left(\frac{1}{3}-\frac{1}{2}\right)$

$= \frac{1}{12}$ ①

c) $\cos x(4\cos^2 x-3)=0$

$\cos x = 0$ or $\cos x = \pm\frac{\sqrt{3}}{2}$ ①

$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}$ ①, $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ ①

2 a) i) $1-\frac{\pi}{4}$ ii) $1-\frac{\pi}{4}-\frac{\pi}{4} = 1-\frac{\pi}{2}$

b) i) $A_1 = \int_1^2 (y-1)^2\,dy$

$= \left[\frac{(y-1)^3}{3}\right]_1^2$

$= \frac{1}{3}(1-0)$

$= \frac{1}{3}\ u^2$

ii) $A_2 = 2-\frac{2}{3}$

$= 1\frac{1}{3}\ u^2$

iii) $Vol = \pi\times 1^2\times 2 - 2\pi\int_0^1 (y-1)^4\,dy$ ① ①

$= 2\pi - 2\pi\left[\frac{(y-1)^5}{5}\right]_0^1$

$= 2\pi - 2\pi\left(0--\frac{1}{5}\right)$

$= 2\pi - \frac{2\pi}{5}$

$= \frac{8\pi}{5}\ u^3$ ①

c) odd function $\Rightarrow$ answer is 0

(3) a) $\cos(\tan x)\sec^2 x$

b) $\int_1^5 f(x)\,dx \doteqdot \frac{1}{2}(0+2.8+6.6+5.6+1.5)$

$= 8.25$

c) i) $x^2 = x+2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$\therefore x = 2$ or -1

ii) $A = \left|\int_{-1}^{2} (x^2 - x - 2)\,dx\right|$ ← (1)

$= \left|\left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-1}^{2}\right|$ ← only if necessary

$=$ (calc.)

$= |-4\frac{1}{2}|$

$= 4\frac{1}{2}\ u^2$ ← (1)

iii) α) $Vol = \pi\int_{-1}^{2} (x+2)^2 - x^4\,dx$ (1)

β) $Vol \doteqdot \pi \times \frac{1\frac{1}{2}}{3}\left[f(-1) + 4f(\tfrac{1}{2}) + f(2)\right]$

$= \frac{\pi}{2}\left(0 + \frac{99}{4} + 0\right)$

$= \frac{99\pi}{8}\ u^3$ } (2)

(4) a) $-\sin\theta \times \frac{1}{\sin\theta} = -1$

b) i) $y = 1 - x - (x-1)^{-1}$

$y' = -1 + (x-1)^{-2}$

$y'' = -2(x-1)^{-3}$

$= \frac{-2}{(x-1)^3}$

ii) S.P.'s when $y' = 0$

$\therefore 1 = \frac{1}{(x-1)^2}$

$\therefore (x-1)^2 = 1$

$\therefore x - 1 = \pm 1$

$\therefore x = 2$ or 0 ← ①

$x = 2$, $y'' < 0 \Rightarrow$ max. T.P. at $(2, -2)$ ①

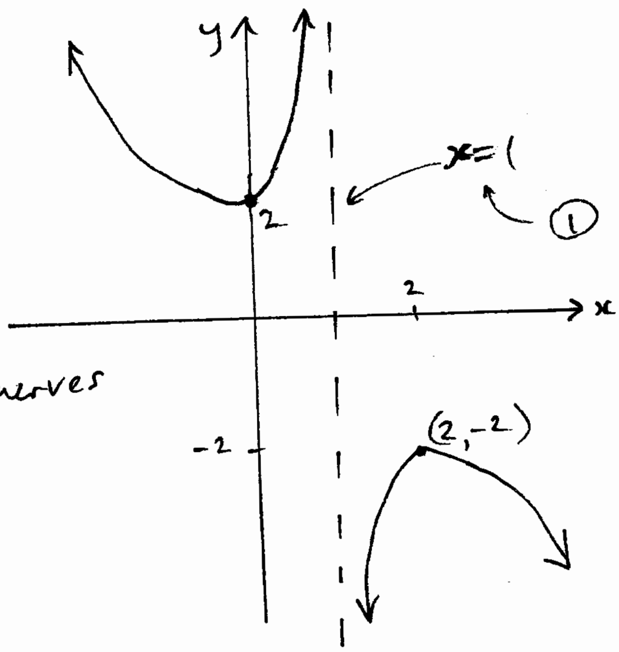
$x = 0$, $y'' > 0 \Rightarrow$ min. T.P. at $(0, 2)$ ①

iii) P. of. I if $y'' = 0$

but $\frac{-2}{(x-1)^3} = 0$ impossible

$\therefore$ no pt of inflexion.

iv)



⑤ a) i) S.P. when $x = -2$ ①

minimum turn. pt.

S.P. when $x = 3$ ①

horizontal pt. of inflexion

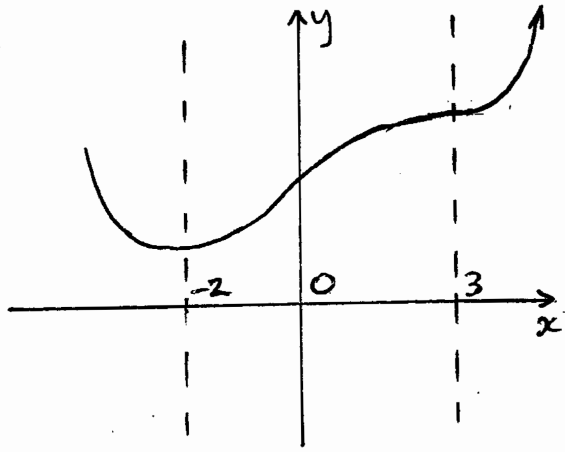
ii) pt. of inflexion, since there is max. pos. gradient when $x = 0$ betw. the two stat. points. ← key point

iii) α) $-2 < x < 3$

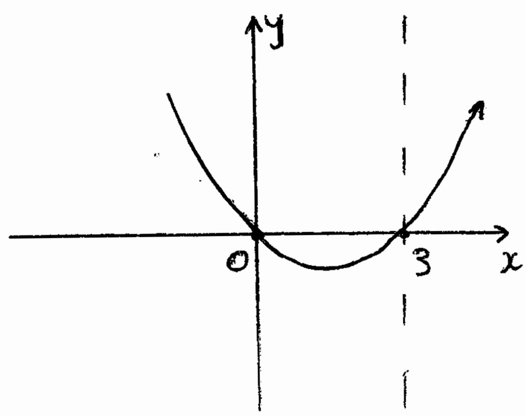
and $x > 3$ (both)

β) $f''(x) < 0$ means that $f'(x)$ is decreasing, ie. $0 < x < 3$.

iv)



v)



b) i) $\frac{d}{dx}(\sec^2 2x) = \frac{d}{dx}\left(\frac{1}{\cos^2 2x}\right)$

$$= \frac{-2\cos 2x \times (-\sin 2x) \times 2}{(\cos^2 2x)^2} \quad \leftarrow (1)$$

$$= \frac{4\cos 2x \sin 2x}{\cos^4 2x}$$

$$= 4\frac{\cos 2x}{\cos 2x} \times \frac{\sin 2x}{\cos 2x} \times \frac{1}{\cos^2 2x}$$

$$= 4\tan 2x \sec^2 2x$$

need to show (1)

ii) $= \frac{1}{4}\left[\sec^2 2x\right]_0^{\pi/3}$

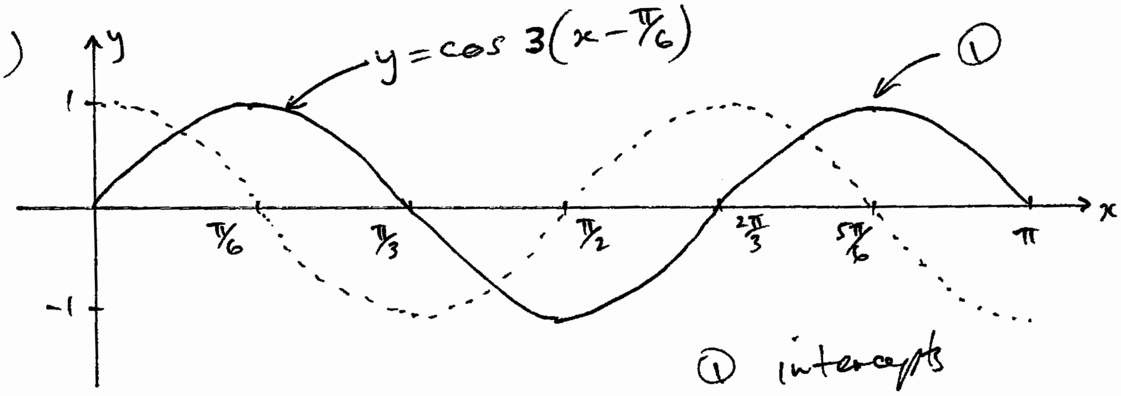
$$= \frac{1}{4}\left[\frac{1}{\cos^2 2x}\right]_0^{\pi/3}$$

$$= \frac{1}{4}\left(\frac{1}{1/4} - \frac{1}{1}\right)$$

$$= \frac{1}{4}(4-1)$$

$$= \frac{3}{4}$$

(6) a) i)



ii) $-1 < k < 0$

b) $d = \sqrt{x^2 + y^2}$

$2x + 2y = p$

$x = \frac{p - 2y}{2}$

$= \sqrt{\left(\frac{p-2y}{2}\right)^2 + y^2}$

$= \sqrt{\frac{p^2 - 4py + 4y^2}{4} + \frac{4y^2}{4}}$

$= \frac{1}{2}\sqrt{p^2 - 4py + 8y^2}$ ← (1) or similar

minimum d when $d' = 0$ (1)

$d' = \frac{1}{2} \times \frac{1}{2}(p^2 - 4py + 8y^2)^{-\frac{1}{2}} \times (16y - 4p) = 0$

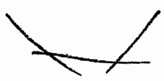
$\therefore \frac{16y - 4p}{4\sqrt{}} = 0$

$\therefore 16y = 4p$ (1)

$\therefore y = p/4$

necessary

y	$p/5$	$p/4$	$p/3$
d'	$-$	0	$+$



$\therefore$ minimum diagonal is proved when $y = p/4$ and $x = \frac{p - \frac{2p}{4}}{2}$

$= \frac{p/2}{2}$

$= p/4$ (1)

$\therefore$ equal sides in rectangle

$\therefore$ square is a rectangle.